Finding the Centre of Infinite Lie Algebra

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Abstract— L reductive lie algebra M is called standard lie algebra , if it is ideal for some Parabolic of subalgebra , this case when M is complete linear algebra are studied by G.B. Gurevich [1, 2] . In work Yu.B. Khakimdzhanov [3] generalization the result of G.B.Gurevich , in condition any radical lie algebra and studied standard subalgebra which has a nilpotent elements (so-called standard nil-subalgebra) .By the same way in this search the results given for infinite lie algebras of infinite matrices (with finite numbers of nonzero elements).All algebras and spaces studied over arbitrary filed characteristic zero.

Index Terms— ideal, linear algebra, nilpotent elements, parabolic of subalgebra, radical lie algebra, reductive lie algebra, standard lie algebra, standard subalgebra, subalgebra.

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1 Introduction

Infinite lie algebra active teaching in the all world, the different between finite and infinite lie algebra ,the finite lie algebra has power classification theorems and general theorem but the infinite lie algebra teaching in some directs ,

a) Infinite lie algebra of finite multiform filed of vectors.
b) Algebra matrices which his elements are algebraic functions under foundation field.
c) Lia algebra operators in Hilbert apace and Banach space.
d) Algebra Katsa – Mudi.

All this directs are studied in this time actively, in this search will studied the Following:

1- The relation between Bounded linearly operators and infinite matrices

$A = (a_{ij},$ when $1 \leq i, j < \infty)$ which has finite numbers of nonzero elements.

2- Writing parabolic subalgebra P of subalgebra M.
3- Given standard nil-subalgebra L in M.
4- Writing the centre C (L) of lie algebra L.

2 Result and discussion

1- let M– lie algebra of infinite matrices $A = (a_{ij},$ when $1 \leq i, j < \infty)$ with finite numbers of nonzero elements . we denoted by e_{ij} to matrix which is in the intersection i – row and j – column exist identity and another places zero , It clear that the matrix e_{ij} where i, j = 1,2,3,…, form a bases to the space M .

Let $h = (h_{ii}) \in$ H, when H – subalgebra all diagonally matrices, then

$[\ h \quad e_{ij}\ ] = (\ h_i - h_j)\ e_{ij} = \propto (h)\ e_{ij}$, for $i \neq j$,

i.e. e_{ij} - Eigen vector for all reform Diff h with Eigen value $\propto(h)$ Linear function $\propto (h) = h_i - h_j$ when $i \neq j$ in H is called roots lie algebra M relatively H , This root in sum time denoted by (i, j) , The set of all roots of lie algebra M relatively H denoted by R , the root (i, j) is called simple, if $j = i + 1$, the set of all simple roots denoted by S , It is clear that every root can written by summation of simple roots with positive coefficient , or negative coefficient equal to 1 or -1 , in the first situation the roots called positively R^+ and the second situation called negatively R^- , then we have $R = R^+ \cup R^-$ In the set R^+defined partially order :

$$(i, j) \geq (k, m) \text{ if } i \leq k, j \geq m.$$

Definition: simple root $\propto \in S$ is called end root for $\beta \in R^+$, if either $\propto = \beta$ or $\beta - \propto \in R^+$, for any $\beta \in R^+$ denoted by S^β the set of all end roots for β .

The subset Δ of system roots R is called closed if , from $\propto, \beta \in \Delta$, $\propto + \beta \in R$ impels $\propto + \beta \in \Delta$, subset Δ of R is called parabolic, if Δ closed and Δ contained R^+,Subalgebra P of M is called parabolic, if it is contain subalgebra $B = H + \sum_{\propto \in R^+} E_\alpha$.

2- let S_1 be subsystem of simple roots system , $\Delta\ (S \setminus S_1)$ is the set of all roots which can write by combination of simple roots from $S \setminus S_1$ and $\Delta_1 = \{\propto \in R : \propto \geq \beta$,for sum $\beta \in S_1\}$, $\Delta = \Delta\ (S \setminus S_1) \cup \Delta_1$.

Subalgebra $T = H + \sum_{\propto \in \Delta\ (S \setminus S_1)} E_\alpha$ and $P = H + \sum_{\propto \in \Delta} E_\alpha$ of M are called respectively reductive subalgebra and parabolic subalgebra which are defined subsystem S_1.

Lemma 1 : let Δ be parabolic subset from R ,then their exist subset $S_1 \subset S$ such that $\Delta = \Delta\ (S \setminus S_1) \cup \Delta_1$.

The next axiom has been shown that subalgebra M contains H given the same properties of regular subalgebra of finite situation.

Lemma 2 : Let subalgebra L from Lie algebra M such that it's normalization N(L) contain H , then their exist $\Delta \subset R$ and $H' \subset H$ such that $L = H' + \sum_{\propto \in \Delta} E_\alpha$.

Now looking for the system of simple roots S , with the following order : $\propto_1 = (1,2)$, $\propto_2 = (2, 3)$, … , $\propto_k = (k, k+1)$, any part $\propto_m, \propto_{m+1}, \ldots, \propto_{m+v}$ this sequence is Called related subset from S , It is clear that any subset from S , we can write by form the union of related component subset from S ,every subset $S_1 \subset S$ defined semi-simple Subalgebra that $F = H_1 + \sum_{\propto \in \Delta} E_\alpha$, when $H_1 = \sum_{\propto \in \Delta} [\ E_\alpha, E_{-\propto}\ ]$, because related component S_1 according to simple subalgebra from partition F , For any subspace $H' \subset H$, contain H', subalgebra $T = H' + \sum_{\propto \in \Delta} E_\alpha$ is reductive subalgebra from M.

Remark: semi-simple lie algebra is called the lie algebra which can write by form summation (my be infinite numbers) simple lie algebras.

Lemma 3: If semi-simple subalgebra $F_0 = H_1 + \sum_{\propto \in \Delta_-} E_\alpha$, contained in F defined subset $S_0 \subset S$. For satisfied that F_0 is an ideal in F it is necessary and enough to show that the subset S_0 is union some of related component from S_1.

Corollary : If radoctiv subalgebra $T_0 = H_0 + , \sum_{\propto \in \Delta_-} E_\alpha$ defined by S_0 ,is not Ideal in $T = H' + \sum_{\propto \in \Delta} E_\alpha$,when $T_0 \subset T$,then either exist related component in S_0 , we can say more (exist related component in S_1 , or H_0 contain one element h such that [h , e_γ] $\neq 0$ for some $\gamma \in (\Delta \setminus \Delta_0)$.

Theorem 1: let S be a system of simple roots in R , let S_1 be subsystem in S , andΔ $(S \setminus S_0)$is the set of all roots which can write by component simple roots of $S \setminus S_0$, And $\Delta_1 = \{\propto \in R, \propto \geq \beta$ for some $\beta \in S_1\}$, then P = H + $\sum_{\propto \in \Delta} E_\alpha$ in M When $\Delta = \Delta(S \setminus S_1) \cup \Delta_1$ is parabolic subalgebra , the inverse , any parabolic subalgebra in M has the same form for some subsystem $S_1 \subset S$.

Proof: it is clear that $\Delta \in R^+$, to proof the theorem in one side it is enough to show that Δ is closed, and since $\Delta(S \setminus S_1)$ and Δ_1 are closed then it is enough to show that $\propto + \beta \in \Delta$, if $\propto \in \Delta(S \setminus S_1)$, $\beta \in \Delta_1$, this result given from definition of subsets Δ_1, $\Delta(S \setminus S_1)$,now proof the inverse of theorem , let P is parabolic subalgebra , then P regular subalgebra by (Lemma 2**)** , it is means that P has form H + $\sum_{\propto \in \Delta} E_\alpha$ for some $\Delta \in R$, and because the subset Δ is parabolic , then by (Lemma 1) $\Delta = \Delta(S \setminus S_1) \cup \Delta_1$ for some $S_1 \subset S$ ■

3- Subalgebra A from Lie algebra M is called standard, if it is ideal of some parabolic subalgebra , Standard subalgebra which is formed from nilpotent elements is called standard nil-subalgebra .

Suppose that the subset $R' \subset R^+$ of different pairwise roots, and $\Delta(R') = \{\beta \in R^+ : \beta \geq \gamma$ for some $\gamma \in R'\}$ then the subspace L = $\sum_{\propto \in \Delta(R')} E_\alpha$ is clear that is standard which is called standard subalgebra defined subsystem R',The next theorem is likewise theorem in finite situation and it's written standard nil-subalgebra , normalization and it has some different proof from proof in [3].

Theorem2:let L–standard nil-subalgebra, then it is define some subsystem $R' \subset R^+$ different pairwise roots .

Theorem 3 : Suppose that L- standard nil-subalgebra defined subsystem $R' \subset R^+$ different pairwise roots ,then parabolic subalgebra N(L) (normalization) defined subsystem $S_\alpha = \cup_{\beta \in R'} S^R$ from system S.

The next theorem is fundamental theorem will be write standard subalgebra in M,with note that formula of theorem is likewise formula theorem in [3] , but the proof is differently.

Theorem 4: Let L – standard nil-subalgebra defined subsystem $R' \subset R^+$ different pairwise roots, T_1- reductive subalgebra ,defined subsyetem $S_1 = S \cap R'$, T_2 - reductive subalgebra , defined subsyetem S_2 , then for any ideal T_0 of algebra T_1 Lie in T_2 the subalgebra A = T_0 + L is standard because parabolic subalgebra N(A) { normalization A in M } homomorphic with N(L) and defined subsystem S_2.

The inverse any standard subalgebra homomorphic with subalgebra A for some subsystem $R' \subset R^+$ and for some ideal T_0.

Proof: we partions the proof to some parts, in the first we proof that A = T_0 + L is standard subalgebra , and for it's enough to show that normalization N(L) subalgebra L is parabolic subalgebra (defined subsystem S_2) , and normalization also A by (theorem3), we denote by P_1 and P_2 parabolic subalgebras defined subsystems S_1, S_2 sequently , and by (theorem 3) we have N(L) = P_2, then $P_i = T_0 + M_i$, $M_i = \sum_{\propto \in \Delta i} E_\alpha$ semi-direct sum, when Δ_i – the set of roots which are greater or equals to some roots from Si:(i=1,2) , since $S_1 \subset S_2$ then $S / S_2 \subset S / S_1$ and so $T_2 \subset T_1$, $P_2 \subset P_1$, $M_1 \subset L \subset M_2$, then we have ;

[N (L), A] = [P_2 , T_0 + L] = [P_2 ,T_0] + [P_2 , L] $\subset$ [P_1 , T_0] + L =[T_1 +M_1 ,T0] + L

= [T_1, T0] + [M_1 , T_0] + L $\subset$ T_0 + [L , T_0] + L =T_0 + L = A ,

Hence we proved that N(L) $\subset$ N(A) ………………. (I),

and it is implies that A is standard subalgebra , Let A – is standard subalgebra ,and by lemma (2) ,we have ; A= $H' + \sum_{\propto \in \Delta} E_\alpha$,For some closed subsystem Δ of the roots system R , Looking for subsystem $\Delta' = \Delta \cap (-\Delta)$ and $\Delta'' = \Delta \setminus \Delta'$ of system Δ, proof that Δ'' net contains negative roots ,suppose the inverse ,that the negative root $-\gamma$ belong to Δ''. By definition of standard A , it has elements $e_{-\gamma}$ and subalgebra A contains element [[e_γ , $e_{-\gamma}$] e_γ] = e_γ , then $-\gamma \in \Delta \cap (-\Delta) = \Delta'$, and it is impossible because $-\gamma \in \Delta''$, It is clear that the subsystem Δ' is closed and with any root contains also all simple roots from it's partions of simple roots , Really if negative root $-\gamma = -\propto_1 - \propto_2 - \ldots - \propto_k$ is belong to Δ' , then with considered closed Δ with additionR^+, the set Δ are contain in any term $-\propto_i$, and Since Δ'' not contains negative roots , then $-\propto_i \in \Delta'$. We obtained Δ' contain all roots which are write by subsystem S' from the system S, this is means that :

$$T_0 = H' + \sum_{\propto \in \Delta'} E_\alpha \quad \ldots\ldots\ldots\ldots \quad (2)$$

Is raductive subalgebra , we proof that Δ' has with any root γ contains all roots $\delta \geq \gamma$, where δ belong to Δ ,this is obtained from which Δ is closed relatively addition in R^+, but $\delta = \gamma + \propto_1 + \propto_2 + \ldots + \propto_k$ ($\propto_i \in S$) is impossibl belong to Δ'(and in the inverse situation ,we obtained that γ belong to Δ') therefore $\delta \in \Delta''$, if we denoted by R' the set of all minimum elements of Δ'' ,then we obtained that Δ'' is contain all elements greater or equal to β for some $\beta \in \Delta''$ it is means that , $L = \sum_{\propto \in \Delta''} E_\alpha$ …………………….. (3),

Is standard nil-subalgebra defended subsystem R' , And from (2) and (3) we have A = T_0 + L (semi - addition straight), we obtained that the normalization N(A) is standard subalgebra A coincide with N(L) , Let parabolic subalgebra N(A) define subsystem $S' \subset S$, we have N(A) = T + M (semi- addition straight), where T = H+ $\sum_{\propto \in \Delta_-} E_\alpha$, and M = $\sum_{\propto \in \Delta_-} E_\alpha$ ($\Delta_\propto$ the set of roots which are larger than or equal for some roots from the set S'),It is clear that $T_0 \subset$ T and also L $\subset$ M . In the inverse situation for some $e_\gamma \in$ L ,this means that $e_\gamma \in$ T, then $e_{-\gamma} \in$ T $\subset$ N(A) and [$e_{-\gamma}$, e_γ] = $h_\gamma \in$ L and it is impossible (because L is contains nilpotent elements) , We have [N(A), L] $\subset$ [N(A) , M] $\subset$ M and [N(A) , L] $\subset$ A , therefore [N(A) , L] $\subset$ A $\cap$ M , observe that A $\cap$ M = $\sum_{\alpha \in \Delta \cap \Delta_-} E_\alpha$= L so we obtained that [N(A) , L] $\subset$ L , from N(A) $\subset$ N(L) ……………..… (4)

From …. (1) and … (4) obtained that N(A) = N(L) , and moreover we received That T_0 contained in T which is equal to T_1 , Now to prove theorem , it is enough to show that T_0 is ideal of T_1 , it is equivalent to that the set $\Delta^{'}$ is closed relative with elements addition from $\Delta(T_1)$, Locking to subsystem $S^{'}$ and S_1 systems of simple roots which are generated $\Delta^{'}$ and $\Delta(T_1)$.

Suppose that T_0 belong to T_2 ($T_0 \subset T_2 \subset T_1$) is not ideal , then their exist relation between the set $S^{'}$ and the set $S^{'}$ which are contained in the set $\bar{S_1}$ of the set and S_1 (look – Corollary of lemma(3). We choose simple roots α , $\beta \in \bar{S_1}$,such that $\alpha \in S^{'}$, $\beta \in S^{'}$ and $\alpha + \beta \in$ R ,then the root $\alpha + \beta$ is impossible be lie in $R^{'}$, so α is edge root for $\alpha + \beta \in R^{'}$ and it is lie in S_2 ,this is impossible because $S^{'} \subset S \setminus S_2$, then $e_{\alpha+\beta}$ is not lie neither in T_0 nor in L, i.e. is not lie in A and in the other side $e_\alpha \in T_1 \subset$A , $e_\beta \in$ B ,then by standard A, [e_α , e_β] = N . $e_{\alpha+\beta} \in$ A (N $\neq$ 0) , is impossible so T_0 is ideal in T_1■

4- let L – standard nil-subalgebra defined subsystem $R^{'} \subset R^+$ (where $R^{'}$ is contain different pairwise roots) ,we put the elements of $R^{'}$ with following order :

(i_1,j_1) , (i_2,j_2) , … , (i_k,j_k) such that ; $i_1 < i_2 < \ldots < i_k$

Theorem 5: If $R^{'}$ is infinite set of elements ,then the centre C (L) algebra L equal to zero and if $R^{'}$ is finite set of elements (k-elements) ,then C (L) = L $\cap$ L_1 , where L_1 – standard nil-subalgebra defined subsystem from one root(j_1,i_k) .

Proof: i) let $R^{'}$ is infinite set of elements ,suppose that C (L) $\neq$ 0 , then their exist non zero element x $\in$ C (L) which that [x , L] = 0 .

Let x = $\sum \lambda_\alpha e_\alpha$,we prove that any next sum also belong to C (L) , i. e. $e_\alpha \in$ C (α) ,for all α , when $\lambda_\alpha \neq 0$,by(lemma5) , for this it is enough that C (L) is normalized by H , i.e. [h ,x] $\in$ C (L) for any h $\in$ H , It is real that

[[h ,x] ,y] = - [[x ,y] ,h] - [[y, h] ,x] = 0 for any y $\in$ L , x $\in$ C(L) and h $\in$ H, Let $e_\alpha \in$ C (L) , finding $\beta \in$ R(L) such that $\alpha + \beta \in$ R , let α = (i,j) , and since $R^{'}$ is contain infinite elements ,so their exist k : $i_k > j$, then if we take m such that m $> i_k$,then we obtained β = (j ,m) $>$ (i_n , i_k) and means that $\beta \in$ R(L) and $\alpha + \beta \in$ R(L) from where [e_α , e_β] $\neq$ 0 , i.e. e_α is not belong to C(L) and we obtained contradiction , hence i) are provided .

ii) let $R^{'}$ is form finite number elements (for example k elements), looking for $e_\alpha \in$ L $\cap$ T_1 , then α=(m ,r), where m $\leq j_1$, r $\geq i_k$, the conduction [e_α , e_γ] $\neq$ 0 for $\gamma \in R^+$ in this case it is equivalence that either γ = (s , m) ,or γ = (r ,t), because s $<$ m , r $<$ t , then γ non greater than or equal to β for any roots $\beta \in R^{'}$,it is means that for all elements e_γ belong to L we have [e_α , e_γ] = 0 , i.e. $e_\alpha \in$ C (L) in the other words L $\cap$ L_1 $\subset$ C (L) ………………………… (5)

Now let $e_\alpha \in$ C(L) and suppose that e_α not belong to L .it is means that α = (m ,r) where either m $\leq j_1$, or r $\leq i_k$, in the first condition root γ = (i_1, m) belong to R(L) and in the second condition γ = (r ,j_k) belong to R(L) , and since $\alpha + \gamma \in$ R(L) then [e_α , e_γ] = N . $e_{\alpha+\gamma} \in$ L (N = ± 1) is impossible because $e_\alpha \in$ C(L) , Then C (L) $\subset$ L $\cap$ L_1 ……… (6)

From (5) and (6) we obtained that C (L) = L $\cap$ L_1 ■

3 CONCLUSION

This paper presents a new results for standard nil-subalgebra ,such that If $R^{'}$ is infinite set of elements ,then the centre C(L) algebra L equal to zero and if $R^{'}$ is finite set of elements (k-elements) , then C (L) = L $\cap$ L_1 , where L_1 – standardnil-subalgebra defined subsystem from one root (j_1,i_k) .

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